

On the couplings of vector mesons in AdS/QCD

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ABSTRACT: We address, in the AdS/CFT context, the issue of the universality of the couplings of the ρ meson to other hadrons. Exploring some models, we find that generically the ρ -dominance prediction $f_\rho g_{\rho HH} = m_\rho^2$ does not hold, and that $g_{\rho HH}$ is not independent of the hadron H . However, we prove that, in any model within the AdS/QCD context, there are two limiting regimes where the $g_{\rho HH}$, along with the couplings of all excited vector mesons as well, become H -independent: (1) when H is created by an operator of large dimension, and (2) when H is a highly-excited hadron. We also find a sector of a particular model where universality for the ρ coupling is exact. Still, in none of these cases need it be true that $f_\rho g_\rho = m_\rho^2$, although we find empirically that the relation does hold approximately (up to a factor of order two) within the models we have studied.

KEYWORDS: Confinement, QCD, AdS-CFT Correspondence.

Contents

1.	Introduction	1
2.	Decomposition in AdS/CFT	3
3.	Universality	8
3.1	ρ dominance and universality	8
3.2	Two examples of approximate universality	10
3.3	Exact universality	17
4.	Conclusions and discussion	19
A.	Review of methodology	20
B.	Review of the hard-wall and the D3/D7 model	22
C.	Proof of decomposition formula	25

1. Introduction

The observed couplings of the octet of vector mesons ($\rho(770)$, $\omega(782)$, $\phi(1020)$, etc.) show an interesting universality, one which is not an obvious consequence of any known QCD mechanism. The ρ decay to $\pi\pi$ gives $g_{\rho\pi\pi}^2/(4\pi) = 2.9$, and isospin-related decays of the ϕ give $g_{\phi K^+K^-}^2/(4\pi) = 3.2$ and $g_{\phi K_L K_S}^2/(4\pi) = 3.5$ [1]; this should be compared with the unrelated process of pion-nucleon scattering, which yields $g_{\rho\pi\pi}g_{\rho NN}/(4\pi) = 2.8$ [2].

In 1960, Sakurai proposed a now-famous conjecture [3], that the ρ meson has a universal coupling to every isospin-carrying hadron. In particular, the “vector meson dominance” conjecture [4–6] sets this universal coupling to the ρ mass-squared divided by the ρ decay constant: $g_\rho = m_\rho^2/f_\rho$. The suggestion is that the form factor of any isospin-carrying hadron H is given by the ρ pole:

$$F(q^2) \approx \frac{f_\rho g_{\rho HH}}{q^2 + m_\rho^2}$$

where f_ρ is the ρ decay constant, m_ρ is its mass, and $g_{\rho HH}$ is a coupling characterizing the interaction of a ρ with the hadron H . The H -independent normalization condition $F(0) = 1$ then fixes g_ρ . Sakurai attempted to implement this idea by formulating the ρ meson as a gauge boson. This approach was influential and inspired much subsequent work.

It seems to us that a modern viewpoint, particularly employing the techniques of AdS/CFT [7], might shed some interesting light on this old issue. One of our motivations in exploring this question is a recent proposal by Son and Stephanov based on dimensional deconstruction [8], which in turn was inspired by the hidden-local symmetry mechanism of Bando *et al.* [9] An interesting aspect of this model is that ρ -dominance for some hadrons is a natural consequence of the properties of wavefunctions in the deconstructed extra dimension. There has been a similar observation in another model based on AdS/CFT [10]. In this theory, the form factors associated with certain conserved global symmetry currents are expressed in terms of only a *finite* sum of poles; these poles correspond to the vector meson states created by the current acting on the vacuum. As in [8], the special properties of the extra-dimensional mode functions play an essential role. Since the theory of [8] is an *ad hoc* model, constructed by hand, it is useful to see that the same mathematics arises in an AdS/CFT context, where the whole structure of the computation, including the extra dimension, arises naturally as the dual picture of a strongly-coupled field theory.

In this paper, we will examine the universality of the ρ 's couplings, and those of other excited vector mesons created by the same current.¹ We find the universality and ρ -dominance conjectures do not hold across entire models, both in that couplings are nonuniversal and do not satisfy $f_\rho g_{\rho HH} = m_\rho^2$. However, we do find that universal couplings for *all* of the vector mesons emerge, model-independently, in two interesting limits. These two examples of approximate universality hold for diagonal couplings of the vector mesons both to hadrons whose interpolating operator has large dimension and to hadrons which have high radial excitation. As we will show, the existence of universal couplings in these limits is a consequence of general properties of the AdS/CFT calculation. We also find an example of exact universality, whose origin is interesting but clearly model-dependent, for the couplings of the ρ to the hadrons within a large sector of a particular model. In this example the universal coupling is of the same order as, but does not equal, m_ρ^2/f_ρ .

The paper is organized as follows. In section 2, we discuss vector meson dominance in the limit of large N and large 't Hooft coupling; a proof of vector meson dominance in this limit, due to Son, is given in appendix C. We will supplement the discussion by examples in two different models, which are reviewed in appendix B. One is the “hard-wall” model, which is used to capture generic features of confining gauge theories. The other model is the D3/D7 system, which has “quarks” in the fundamental representation, and associated strongly-coupled “quarkonium” bound states. In section 3, we briefly discuss ρ -dominance and how it motivates the study of coupling universality. Then, we lay out the various types of universality that we have explored, illustrating them with examples from the hard-wall and D3/D7 models. Section 4 contains some concluding remarks. A review of the basic methodology needed from the AdS/CFT dictionary can be found in appendix A.

¹We will generically call the lowest-mass state created by a conserved current acting on the vacuum the “ ρ ”, at the risk of some confusion.

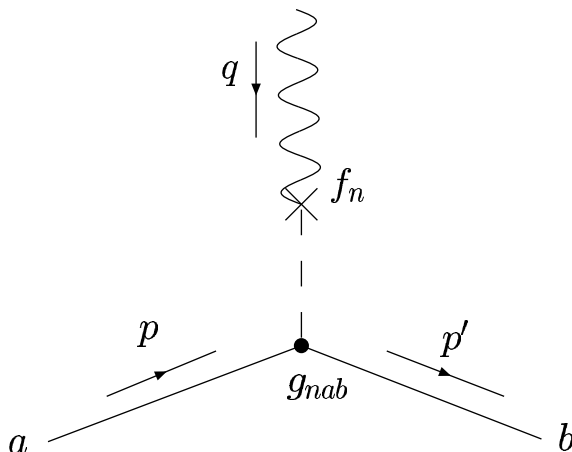


Figure 1: Decomposition of the form factor into a sum over hadron states, as in eq. (2.1).

2. Decomposition in AdS/CFT

In the literature on hadronic physics, it is often assumed that the form factor for a hadron associated with a conserved spin-one current can be written as a sum over vector-meson poles. While this is not in general justified, it is believed to be true at large N . The main goal of this section is to argue this is indeed always true in AdS/CFT contexts, when both the number of colors N and the 't Hooft coupling $\lambda = g^2 N$ are large. Here g is the Yang-Mills coupling.

In particular, we claim (and sketch a proof, due to Son [11], in appendix C) that in confining gauge theories with a supergravity dual,²

$$F_{ab}(q^2) = \sum_n \frac{f_n g_{nab}}{q^2 + m_n^2}, \tag{2.1}$$

as illustrated in Fig. 1. Here f_n denotes the hadron decay constant of the n -th vector hadron state, g_{nab} its coupling to an incoming and outgoing hadron, and m_n its mass.

Before we begin, we need to review how the form factor is computed on the gravity side of AdS/CFT. Leaving the details to appendix A, we cover only what is needed in this section. According to the AdS/CFT duality, a local conserved spin-one current in the gauge theory is dual to a *non-normalizable* mode of a gauge field in the asymptotically-AdS₅ space. Meanwhile, the spin-one hadron state created by the current operator corresponds to a *normalizable* mode of the same gauge field. Now, the form factor is computed by the overlap integral, eq. (A.5), of a non-normalizable mode and two normalizable modes, corresponding to the vector current, an incoming hadron, and an outgoing hadron. The three hadron

²There can be many form factors depending on whether the current is conserved, and on the spins of the hadrons. In this paper, we only deal with conserved currents, whose matrix elements between scalar hadrons have only one form factor. For vector hadrons, there are three form factors: electric F_e , magnetic F_m and quadrupole F_Q . As observed by Son and Stephanov [12] and discussed in [10], the large 't Hooft coupling limit implies $F_e = F_m$ and $F_Q = 0$. Therefore, our discussion focusing on only one form factor is justified.

coupling, in which the current is replaced by a spin-one hadron created by that current, is obtained by the *same integral* except for the replacement of the non-normalizable mode of the five-dimensional gauge field with a normalizable one, as in eq. (A.8).³ Therefore, the decomposition (2.1), if true, must be derived simply from a relationship between the normalizable and non-normalizable modes.

The required relationship is the following. Let's consider a spin- J ($J \leq 2$) field living in the asymptotically AdS₅ space (which we will assume is embedded in a d -dimensional asymptotically- AdS₅ \times W space, with W a compact manifold of dimension $d - 5$.) The mode of the field with momentum q^μ may be written $C_{\mu_1 \dots \mu_J}(q) = \epsilon_{\mu_1 \dots \mu_J} e^{iq \cdot x} \chi(q^2, z)$, where z is the five dimensional radial coordinate defined in appendix A, and μ runs from 0 to 3. The normalizable mode is $\phi_n(z) \propto \chi(-m_n^2, z)$ at $q^2 = -m_n^2$, and the non-normalizable mode $\psi(q^2, z) \equiv \chi(q^2, z)$ for arbitrary q^2 can be written as

$$\psi(q^2, z) = \sum_n \frac{f_n \phi_n(z)}{q^2 + m_n^2} \tag{2.2}$$

$$f_n = \lim_{z \rightarrow 0} \frac{V(z)}{g_d} \left(\frac{R}{z} \right) \partial_z \phi_n(z). \tag{2.3}$$

Here g_d is a d -dimensional coupling constant; its precise form depends on the current and the theory under study. Meanwhile $R = \lambda^{1/4} \alpha'^{1/2}$ is the AdS₅ curvature radius, and $V(z)$ is the volume of W at z . Substitution of eq. (2.2) into (A.5) and using (A.8) yields eq. (2.1). A proof of (2.2) and (2.3) for spin-one currents is given in appendix C. Additional details about our notation are given in appendix A.

It will be useful below to recall the scaling properties of the f_n in models with supergravity duals. The decay constant f_n of a spin-one hadron is defined by

$$\langle 0 | \mathcal{J}^\mu(x=0) | n, p, \epsilon \rangle = f_n \epsilon_\nu,$$

where $|0\rangle$ is the vacuum of the theory, and $|n, p, \epsilon\rangle$ is the spin-one hadron state with mass m_n , momentum p and polarization ϵ_μ created by the conserved current operator \mathcal{J}^μ . The f_n and m_n are constrained by the fact that the two-point correlation function of a conserved current can be written

$$\langle \mathcal{J}_\mu(q) \mathcal{J}_\nu(-q) \rangle \sim q^2 \ln q^2 \left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) = (q^2 \eta_{\mu\nu} - q_\mu q_\nu) \sum_n \frac{|f_n|^2}{m_n^2 (q^2 + m_n^2)}. \tag{2.4}$$

If $m_n \sim n^p$ at large n , then the $\log q^2$ behavior requires $f_n \sim n^{2p-1/2}$. The supergravity limit has $p = 1$, so $f_n \sim n^{3/2}$ for a conserved current. Similarly, the energy momentum tensor has $f_n \sim n^{5/2}$ since its two-point function goes as $q^4 \ln q^2$. We know of no similarly useful constraints on the three-hadron couplings g_{nab} , unless all three hadrons are highly excited, a case we will not discuss.

³For spin-two currents, the same statements hold with all spin-one currents and hadrons replaced with spin-two, and with the five-dimensional gauge field replaced by the five-dimensional graviton.

Examples

We now illustrate the above formalism through a few examples. Our computation will be mostly focused on, first, showing the decomposition (2.1) explicitly in two exactly solvable models, and second, verifying the formula for the hadron decay constant f_n (2.3). The computed f_n 's will be also useful in comparing the examples in future sections with previous work.

The first theory we consider is the hard-wall model, which effectively models the behavior of a large class of confining theories in the large λ limit. On the gravity side, the theory is simply given by an $\text{AdS}_5 \times S^5$ background cut off by a wall at finite radius, where boundary condition for mode functions are imposed. We leave the detailed discussion of this theory to the original literature [13–15], but the list of mode functions that we will use can be found in appendix B.

The other model that we use is the flavor-non-singlet sector of a system of N D3-branes and N_f D7 branes. This model has a distinctive feature; the theory has QCD-like mesons, bound states built from matter in the fundamental representation of $SU(N)$. The meson spectrum has been largely worked out. We again refer to appendix B for a brief introduction to the theory and for the mode functions; the reader may wish to consult the original literature [16–18, 10] for a more detailed description of the theory.

Hard-wall model. In the hard-wall model, we recall the non-normalizable (B.3) and normalizable mode (B.1) for a gauge field in this theory,

$$\begin{aligned} \psi(q, z) &\approx \frac{1}{g_{10}} qz K_1(qz) \quad (\text{non-normalizable mode}), \\ \phi_n(z) &= \frac{\sqrt{2}\Lambda z J_1(\zeta_{0,n}\Lambda z)}{\pi^{\frac{3}{2}} R^3 J_1(\zeta_{0,n})} \quad (\text{normalizable mode}). \end{aligned}$$

where $\zeta_{\nu,n}$ is the n^{th} zero of J_ν , and the approximation in the first equation is that $q \gg \Lambda$. As explained in appendices A and B, the canonical normalization of the non-normalizable mode requires division by $g_{10} = \kappa/R$, the effective coupling constant for a spin-one mode in the hard-wall model. These two functions are related by a mathematical identity,

$$q^\nu K_\nu(qx) = \int_0^\infty dm \frac{m^{\nu+1} J_\nu(mx)}{q^2 + m^2}. \quad (2.5)$$

For $\nu = 1$, this formula is of the same form as eq. (2.2), except that the sum over states has been replaced with an integral over a continuous spectrum. The reason for this is that in constructing the non-normalizable mode as in (B.3) we ignored the boundary condition on the wall, which leads us to a continuous spectrum. This spectrum approximates the true discrete result in the limit of high-mass states ($n \gg 1$) or equivalently in the limit of small confinement scale Λ .

Comparing eq. (2.2) with (2.5), we see f_n is given simply by m_n^2 divided by the normalization coefficient of the normalizable mode.

$$f_n = \frac{dm_n}{dn} m_n^2 \left(\frac{\sqrt{2}\Lambda}{\pi^{\frac{3}{2}} R^3 J_1(m_n/\Lambda)} \right)^{-1} \left(\frac{R}{\kappa} \right) \approx \frac{\pi^2}{2\sqrt{2}} n^2 J_1(\zeta_{0;n}) \Lambda^2 N \approx \frac{\pi}{2} n^{3/2} \Lambda^2 N. \quad (2.6)$$

The powers of N and Λ in this result are fixed on general grounds by N -counting and dimensional analysis. The $n^{3/2}$ scaling is required by ultraviolet conformal invariance, as explained earlier.

Meanwhile eq. (2.3) applied directly to the normalizable mode ϕ_n gives

$$f_n = \lim_{z \rightarrow 0} \frac{R}{\kappa} \cdot \frac{\pi^3 R^6}{z} \cdot \partial_z \frac{\sqrt{2} \Lambda z J_1(\zeta_{0;n} \Lambda z)}{\pi^{\frac{3}{2}} R^3 J_1(\zeta_{0;n})} = \frac{N \Lambda^2 \zeta_{0;n}}{\sqrt{2} \pi J_1(\zeta_{0;n})} \approx \frac{\pi}{2} n^{3/2} \Lambda^2 N.$$

As we noted earlier, the discrepancy between these equations arises from the fact that eq. (2.5) is exact only in the strictly conformal limit $\Lambda \rightarrow 0$; the reader is invited to check that the discrepancy is removed when the exact form of the non-normalizable mode, eq. (B.3), is used. As required, the two expressions match in the large n limit.

For future comparison, we also compute the ratio between m_n^2 and f_n ,

$$\frac{m_n^2}{f_n} = \frac{1}{2} \zeta_{0;n} J_1(\zeta_{0;n}) \frac{(2\pi)\sqrt{2}}{N}. \quad (2.7)$$

In particular, for the ρ ($n = 0$),

$$\frac{m_\rho^2}{f_\rho} = 0.624 \frac{(2\pi)\sqrt{2}}{N}. \quad (2.8)$$

The extension to the energy-momentum tensor is straightforward. It corresponds to the $\nu = 2$ case in eq. (2.5) and the decomposition is explicit. Once again, we also read off the decay constant of a spin two hadron,

$$f_n = \frac{dm_n}{dn} \left(\frac{m_n^3}{2} \right) \left(\frac{\sqrt{2} \kappa \Lambda}{\pi^{\frac{3}{2}} R^4 J_2(m_n/\Lambda)} \right)^{-1} \approx \frac{\pi^3}{4\sqrt{2}} n^3 J_2(n\pi) \Lambda^3 N \sim \frac{\pi^2}{4} n^{5/2} \Lambda^3 N.$$

Eq. (2.3) gives

$$f_n = \lim_{z \rightarrow 0} \frac{1}{\kappa} \cdot \frac{\pi^3 R^6}{z} \partial_z \frac{\sqrt{2} \Lambda J_2(\zeta_{1;n} z)}{\pi^{\frac{3}{2}} R^2 J_2(\zeta_{1;n})} = \frac{\zeta_{1;n}^2 \Lambda^3 N}{2\sqrt{2} \pi J_2(\zeta_{1;n})} \approx \frac{\pi^2}{4} n^{5/2} \Lambda^3 N.$$

Again we observe coincidence in the conformal limit.

D3/D7 model. Let us now turn to the D3/D7 system. It has been shown that the decomposition (2.1) is explicit in this case [10]. Hence, we will check only the formulas for f_n (2.3) in the following ways: first, we compute f_n for the cases where the form factors are known explicitly, and second, we compare the result with the one from Eq (2.3). We will also read off f_n from the form factors using different external hadrons; as we will see, the f_n 's are independent of the external hadrons, as they should be.

Using the metric and the mode functions given in appendix B, the coupling constant is obtained by an overlap integral for the type I modes and the vector mode, expressed as

$$g_{n,n_1,n_2}^\ell = g_8 \frac{L^2}{2} (2\pi^2) \int_0^1 \frac{dv}{v^2} \phi_{0,n}^{II}(v) \phi_{\ell,n_1}^I(v) \phi_{\ell,n_2}^I(v), \quad (2.9)$$

where g_8 is the Yang-Mills coupling of the eight dimensional D7 worldvolume theory and L is the distance between the D7 and D3 branes, which sets the quark mass $m_Q = L/\alpha'$. The typical meson mass scale is set by $m_h = L/R^2 = m_Q/\sqrt{\lambda}$. We compute a special case of the vector $(0, n)$ – scalar $(1, n_2)$ – scalar $(1, 0)$ overlap integral,

$$g_{n,0,n_2}^{\ell=1} = (-1)^{n+n_2+1} \frac{(2\pi)}{\sqrt{N}} \sqrt{\frac{3}{2(n+1)(n+2)(2n+3)(2n_2+3)}} \\ \times [n_2(n_2+2)\delta_{n,n_2-1} - (2n_2+3)\delta_{n,n_2} - (n_2+1)(n_2+3)\delta_{n,n_2+1}]. \quad (2.10)$$

Comparing this with the form factor computed in [10], we obtain the decay constant f_n of the vector meson:⁴

$$f_n = (-1)^n \frac{m_h^2 \sqrt{N}}{(2\pi)} \sqrt{8(n+1)(n+2)(2n+3)} \sim m_h^2 n^{3/2} \sqrt{N}, \quad (2.11)$$

where we used $(L^2/R^4)(R^2/g_8) = m_h^2 \sqrt{2N}/(2\pi)^2$. The $n^{3/2}$ scaling for large n is required by conformal invariance, while the powers of N and m_h are fixed on general grounds by N -counting and dimensional analysis.

We may now cross-check this result. The type II normalizable mode with $\ell = 0$ is

$$\phi_n^{II}(\varrho) = \frac{C_{0n}^{II}/R^2}{(1+\varrho^2)^{n+1}} F(-n, -1-n; 2; -\varrho^2) \\ = \frac{C_{0n}^{II}}{R^2} \frac{(-1)^n}{\varrho^2} + O\left(\frac{1}{\varrho^3}\right).$$

Using this and eq. (2.3), we obtain

$$f_n = (2\pi^2/g_8)\rho^3 \partial_\rho \phi_n^{II}(\rho)|_{\rho \rightarrow \infty} \\ = (-1)^n (2\pi)^2 (L^2/g_8 R^2) C_{0n}^{II} \\ = (-1)^n \frac{m_h^2 \sqrt{N}}{(2\pi)} \sqrt{8(n+1)(n+2)(2n+3)}, \quad (2.12)$$

which is exactly eq. (2.11). Note that (in analogy to eq. (2.7))

$$\frac{m_n^2}{f_n} = (-1)^n \sqrt{\frac{2(n+1)(n+2)}{2n+3}} \frac{(2\pi)}{\sqrt{N}} \quad (2.13)$$

and for the ρ ($n = 0$)

$$\frac{m_\rho^2}{f_\rho} = \frac{2}{\sqrt{3}} \frac{(2\pi)}{\sqrt{N}}. \quad (2.14)$$

For comparison, we compute the three vector hadron coupling also. It is given by almost the same integral as (2.9), except that $\phi_{\ell,n_i}^I \rightarrow \phi_{\ell,n_i}^{II}$; also the metric factor changes accordingly:

$$g_{n,n_1,n_2}^\ell = g_8 \frac{R^4}{2} (2\pi^2) \int_0^1 dv \left(\frac{1-v}{v}\right) \phi_{0,n}^{II}(v) \phi_{\ell,n_1}^{II}(v) \phi_{\ell,n_2}^{II}(v). \quad (2.15)$$

⁴Note that we have used a slightly different overall normalization in this paper compared to [10]; f_n and g_n both differ by a factor of $2\pi^2$, the volume of a unit 3-sphere. The change cancels in form factors where only $f_n g_n$ appears.

Now the vector $(0, n)$ – vector $(0, n_2)$ – vector $(0, 0)$ overlap integral is

$$g_{n,0,n_2}^{\ell=0} = (-1)^{n+n_2+1} \frac{(2\pi)}{\sqrt{N}} \sqrt{\frac{3(n_2+1)(n_2+2)}{(n+1)(n+2)(2n+3)(2n_2+3)}} \times [n_2\delta_{n,n_2-1} - (2n_2+3)\delta_{n,n_2} + (n_2+3)\delta_{n,n_2+1}] \quad (2.16)$$

(which is actually symmetric under $n \leftrightarrow n_2$, despite appearances.) Comparing eq. (2.16) with the matrix element obtained in [10], we get the same result for the decay constant f_n as we did in eq. (2.11), as of course we should.

3. Universality

3.1 ρ dominance and universality

In the limit of large N and large λ , as shown in the previous section, vector meson dominance is exact, in a sense of the decomposition (2.1). However, ρ dominance cannot be exact, on completely general grounds, at any N or λ , in a theory in which conformal invariance is exact (or violated only by logarithmic running) in the ultraviolet. In particular, dominance of form factors by the ρ pole simply cannot be true in general at large q^2 . Conformal invariance in the ultraviolet requires the form factor of a spin-zero hadron $|a\rangle$ created by an operator of dimension Δ , must fall as $1/q^{2(\Delta-1)}$. More generally

$$\lim_{q^2 \rightarrow \infty} F_{ab}(q^2) \sim \frac{1}{q^{2k}}, \quad (3.1)$$

where k depends on the spin and twist of the operator creating the a hadron. For example, $k = 2$ for the form factor of the ρ , and for any spin-one hadron created by a conserved current. This behavior, under the assumption that “vector meson dominance” is true, requires a conspiracy between at least k poles.

Consequently the question of ρ dominance can only be relevant at small q^2 , *i.e.*, the issue is whether

$$F_{aa}(q) \approx \frac{f_\rho g_{\rho aa}}{q^2 + m_\rho^2},$$

to some rough approximation, for small $|q^2| \lesssim m_\rho^2$. Since $F(0) = 1$, this, if true, would imply $f_\rho g_{\rho aa} \approx m_\rho^2$, independent of a . Strong ρ dominance implies a universal coupling, and sets its value. But we will see this is not generally true in AdS/CFT.

However, it is logically possible to have exactly or approximately universal couplings without ρ dominance, and in this case the universal coupling need not equal its special value m_ρ^2/f_ρ . We will see this happens in some sectors of AdS/CFT.

Interestingly, the most general situation seems to be that ρ couplings in AdS/CFT contexts, though nonuniversal, tend to lie in a rather narrow range, not varying by more than a factor of two from m_ρ^2/f_ρ . This, combined with the structure of the spectrum, leads to an *apparent* form of ρ dominance that can hold even when the ρ pole is not a dominant contributor to the form factor at small q^2 . We will consider this issue in a later paper [27].

Let us begin the exploration of this issue with some examples that dispel any hope of completely universal couplings.

Examples

Here we compute g_{n00} for some lowest-lying hadrons in the hard-wall and D3/D7 model, comparing in each case the spin-one form factors for scalar and vector hadrons, and finding they are not, in fact, universal. Moreover, we will also find that the ρ meson pole is not always approximately dominant at small q^2 ; in fact, it is possible for small n that $f_n g_{n00} \geq f_0 g_{000}$. (It can even happen that $f_n g_{naa}/m_n^2 \geq f_0 g_{0aa}/m_0^2$; we will explore this in a later publication [27].)

D3/D7 model. In the case of the D3/D7 system, the flavor form factor is easily computed for the lowest-lying mesons within the type I scalar and the type II vector sectors in B; note the latter is the ρ itself. They are

$$F_{0,0}^I(q) = \frac{6m_h^2}{q^2 + m_0^2} + \frac{6m_h^2}{q^2 + m_1^2} \rightarrow \frac{12}{q^2}, \quad q^2 \rightarrow \infty,$$

$$F_{0,0}^{II}(q) = \frac{12m_h^2}{q^2 + m_0^2} - \frac{12m_h^2}{q^2 + m_1^2} \rightarrow \frac{12}{q^4}, \quad q^2 \rightarrow \infty,$$

where $m_n^2 = 4m_h^2(n+1)(n+2)$. Here the sums over poles actually truncate, but in the first case the truncation occurs at $n = \Delta = 2$ rather than at the minimally required $n = \Delta - 1 = 1$. From this simple example we immediately learn that dominance by the ρ is only approximate even for $\Delta = 2$ scalars and vectors; in both cases the contribution of the first excited spin-one hadron is only slightly smaller than that of the ρ , since $m_1^2/m_0^2 = 3$. For the scalar, the ρ contributes about 75% of $F(q^2 \rightarrow 0)$; in particular $f_0 g_{000} = \frac{3}{4} m_0^2$. Moreover, for the scalar $\Delta = 2$ hadron, where the ρ pole *could* have sufficed to satisfy the power law $F(q^2) \rightarrow \# / q^2$ as $q^2 \rightarrow \infty$, it nonetheless did not; thus we see that a natural guess, that conformal invariance might imply that $f_n g_{naa} \ll f_0 g_{0aa}$ for $n \geq \Delta$, is wrong, although it happens to be correct for the form factor of the ρ . Finally, universality of the ρ coupling fails; since f_0 is independent of the external hadron, the first terms of the two form factors imply the corresponding ρ couplings differ by a factor of two.

Hard-wall model In the hard-wall model, the ground-state scalar hadron created by a $\Delta = 2$ operator has a form factor with $|f_n g_{n00}|$ peaking at $n = 2$, with $f_2 g_{200}/f_0 g_{000} \approx 5.86$. Because $m_2^2 = 12.9 m_0^2$, the ρ and second-excited state make comparable contributions at small q^2 : $f_2 g_{200}/m_2^2 \approx 0.32$ and $f_0 g_{000}/m_0^2 \approx 0.72$, while other states, including $n = 1$, make much smaller contributions, of varying sign. Thus we again do not find strong ρ -dominance, though the ρ is still the most important contribution at small q^2 , only slightly less important than in the D3/D7 case. The form factor of the ρ itself, on the other hand, has an interestingly similarity to that of the ρ of the D3/D7 model; $f_n g_{n00}/f_0 g_{00}$ is 1.00, -1.02 and 0.02 for $n = 0, 1, 2$ respectively, with the remainder extremely small. (We will comment on this similarity below.) However, the the f_n and m_n differ in the two models, so the $g_{n\rho\rho}$ do as well. The hard-wall model has $m_1^2/m_0^2 = \zeta_{1,1}^2/\zeta_{1,0}^2 \approx 5.26$, compared to

4 in the D3/D7 model, and $f_1/f_0 = -3.50$ in the hard-wall model and $\sqrt{7}$ in the D3/D7 model. The deviation of $f_0 g_{000}/m_0^2$ from 1 is a bit smaller than in the D3/D7 model, about 24%. Finally the ratio of g_{000} for spin-zero hadrons of $\Delta = 2$ to g_{000} (the ρ self-coupling) is 0.581, compared to 1/2 in the D3/D7 model.

In summary, just looking at a pair of simple states in two models, we see both ρ dominance and coupling universality violated at order one, although not by orders of magnitude. Later we will see cases where ρ dominance is a much worse approximation, though the ρ couplings will still not vary over a large range.

3.2 Two examples of approximate universality

Despite the absence of ρ -coupling universality, we can show that, at large 't Hooft coupling, there are two limits in which coupling universality arises, on very general grounds. Indeed, in these regimes the couplings of *all* of the vector mesons, and indeed the entire form factor, becomes universal, as we observed already in [10]. Both examples stem from the simplification of mode functions in the associated limit.

The first case concerns hadrons created by an operator with large conformal dimension. Under AdS/CFT duality, an operator with conformal dimension Δ corresponds to a five-dimensional field whose mass is $m \approx \Delta/R$, so large Δ corresponds to a heavy particle in five dimensions. Gravity tends to pull particles down to the minimal possible AdS radius, or more precisely, to the minimum of some effective potential due to gravity and other effects. As always, a light particle will have a rather diffuse wave function spread out around the minimum of this potential, while a heavy particle will have a wave function highly concentrated at the potential's minimum. For example, in the duals of confining theories, as captured in part by the hard-wall model, the normalizable mode corresponding to a hadron created by an operator with $\Delta \gg 1$ generically is localized near the wall, where g_{00} is minimized. (This fact was used in obtaining a string theory for high- Δ hadrons in string backgrounds dual to a confining gauge theories [19].) In the D3/D7 system, the normalizable modes of the flavor-charged meson-like states localize at $\varrho = 1$, where ϱ , which runs from 0 to infinity, is the radial coordinate on the D7 branes introduced in appendix B. Therefore, if we take the limit that $\Delta \gg 1$ for the hadrons a and b , the coupling of a vector hadron $|n\rangle$ to these hadrons will only depend on the wave function ϕ_n of the vector hadron, evaluated at the minimum of the potential for the field associated to hadrons a and b . The effective potential does not depend on the particle's mass, Δ , so the position of its minimum is Δ -independent as well as a -independent. The resulting overlap integral is then easily approximated and depends only on n . Consequently, because of the decomposition (2.1), the entire form factor $F(q^2)$ becomes independent of a and Δ as $\Delta \rightarrow \infty$. In general, however, the convergence to the universal form factor and couplings may be very slow.

The second case of universal couplings appears when $a = b$ and the hadron a is a very highly excited state. In this case, its mode function oscillates rapidly with radius. If the oscillation wavelength is sufficiently short, while the mode function for the vector

hadron $|n\rangle$ is slowly varying, then we can approximate the latter as constant in any region of integration, replacing the product $|\phi_a|^2$ by its average, *i.e.*, half its maximum. More precisely and more generally, using the notations in appendix A, we want to compute

$$g_{naa} = g_d \int_0^{z_{max}} dz \mu \phi_n \phi_a^2 \tag{3.2}$$

where $\mu = R^{5-2(S+J)} V(z) e^{(5-2S-2J)A(z)} / z^{5-2(S+J)}$ is the metric factor and g_d is the d -dimensional coupling. (Recall $S = 1$ is the spin of the hadron $|n\rangle$ and J is the spin of the hadron $|a\rangle$.) In the limit that $\phi_a(z)$ oscillates rapidly, we can use the WKB approximation,

$$\phi_a(z) \approx \text{Re} N_a(z) e^{i\varphi_a(z)}. \tag{3.3}$$

and can average the oscillations to obtain

$$g_{naa} = g_d \int_0^{z_{max}} dz \mu \phi_n \phi_a^2 \approx \frac{g_d}{2} \int_0^{z_{max}} dz \phi_n(z) \mu |N_a(z)|^2.$$

We will now show that μN_a^2 has no leading dependence on a , J , or the conformal dimension Δ of the interpolating operator when the excitation number a gets large. Consequently, in this limit, g_{naa} is universal.

Let's first consider the case where $|a\rangle$ has spin zero. The mode function $\phi_a(z)$ satisfies the Klein-Gordon equation in the asymptotically $\text{AdS}_5 \times W$ space, which is given in appendix A,

$$-\frac{1}{\sqrt{g}}(g^{zz} \sqrt{g} \phi_a(z)')' - m_a^2 g^{00} \phi_a(z) + \tilde{m}^2 \phi_a(z) = 0, \tag{3.4}$$

where $g = (R e^{A(z)} z^{-1})^5 V(z)$, and \tilde{m} is the *five dimensional* mass which corresponds to the conformal dimension $\Delta \approx \tilde{m} R$. Eq. (3.4) can be transformed to a Schrödinger equation. When $\psi_a(z) = (g^{zz} \sqrt{g})^{1/2} \phi_a(z)$, we have

$$-\psi_a'' + U(z) \psi_a = m_a^2 \psi_a, \tag{3.5}$$

$$U(z) = \frac{15}{4z^2} + \frac{\tilde{m}^2 R^2 e^{2A(z)}}{z^2} - \frac{3}{16} \left[\frac{2V'}{V} - \frac{(e^{2A(z)} z^{-2})'}{e^{2A(z)} z^{-2}} \right]^2 + \frac{(V^2)''}{4V^2} + \frac{3(e^{2A(z)} z^{-2})''}{4e^{2A(z)} z^{-2}}$$

The approximate solution is given by

$$\psi_a(z) \approx \frac{\text{const.}}{[m_a^2 - U(z)]^{1/4}} \exp \left[i \int^z dy \sqrt{m_a^2 - U(y)} \right]$$

with the quantization condition

$$\int_0^{z_{max}} dz \sqrt{m_a^2 - U(z)} = \left(a + \frac{1}{2} \right) \pi \quad (a = 0, 1, 2, \dots). \tag{3.6}$$

While $U(z)$ is fixed by the metric and \tilde{m} , the mass m_a can be arbitrarily large as we increase a . Thus, we can take the limit that m_a is so large that $U(z)$ is negligible except at small values of z (where the contribution to the g_{naa} integral is small.) In this limit, $\sqrt{m_a^2 - U(z)} \approx m_a$, and we obtain $m_a \approx \pi a / z_{max}$ from the quantization (3.6). Also, in

this limit, $\psi_a(z)$ can be further approximated as

$$\psi_a(z) \approx \tilde{N} \exp(im_a z),$$

where \tilde{N} is a constant which can be determined by the normalization condition (A.2). Therefore, $N_a(z)$ and $\varphi_a(z)$ in eq. (3.3) are completely fixed: $N_a(z) \approx \tilde{N}(g^{zz}\sqrt{g})^{-1/2}$, which is independent of a and Δ , and $\varphi_a(z) \approx m_a z$. From this it follows that g_{naa} is independent of a and Δ .

The approximation $\sqrt{m_a^2 - U(z)} \approx m_a$ breaks down when $U(z)$ becomes of order m_a , which generally occurs in the small- z region. Here the term in $U(z)$ depending on the conformal dimension in Eq. (3.5), $15/4z^2 + \tilde{m}^2 R^2 e^{2A(z)}/z^2 \approx (4\Delta^2 e^{2A(z)} + 15)/4z^2$, diverges. Since the space is nearly $\text{AdS}_5 \times W$ in the region, this term can be neglected for

$$\frac{\Delta^2 + \frac{15}{4}}{z^2} \ll m_a^2 \quad \Leftrightarrow \quad z \gg \frac{\sqrt{\Delta^2 + \frac{15}{4}}}{a} z_{max}.$$

In other words, the region where our approximation is not valid will expand as Δ gets large with a fixed, and this calculation is valid only for $\Delta \ll a$. But in the small z region, the wave function matches on to a known $z^{-\Delta}$ power law, and so the contribution of the small z region to any calculation is generally small, especially at large Δ .

Our discussion so far can be easily generalized to the case where $|a\rangle$ has spin J . In the WKB approximation (3.3), $\varphi_a \approx m_a z$ and

$$N_a(z) \approx (z/R)^{(3-2J)/2} \frac{e^{(J-3/2)A(z)}}{\sqrt{z_{max} V(z)}}. \tag{3.7}$$

and consequently

$$\mu N_a(z)^2 \approx (R e^{A(z)}/z)^{2(1-S)}$$

Thus the integrand in (3.2) is independent of a , Δ and J for $a \gg \Delta$. Consequently, as before, g_{naa}^Δ depends only on n in this limit, and so all hadrons of any J with $a \gg \Delta$ have a universal form factor.

Examples

Large dimension. First we consider the case of $\Delta \gg a$, or in the notation that we have used for the examples, $\Delta \gg n_1, n_2$. In the hard-wall model, as we mentioned earlier, a normalizable mode associated to a hadron $|n_1\rangle$ localizes at the wall, $z = z_{max} = 1/\Lambda$. Thus, for $n \ll n_1, n_2 \ll \Delta$, the first kind of universal vector hadron coupling is given by the value of the normalizable mode $\phi_n(z_{max})$:

$$g_{n,n_1,n_2}^\Delta \xrightarrow{\Delta \rightarrow \infty} \delta_{n_1,n_2} \frac{(2\pi)\sqrt{2}}{N}, \tag{3.8}$$

where we denoted the conformal dimension of the operator creating the other two hadrons by Δ . Note $g_{n,n_1,n_2}^\Delta \sim 1/N$ is consistent with N -counting analysis. That this is Δ - and n_1 -independent is as we expected. That it is n -independent appears to be an accident of

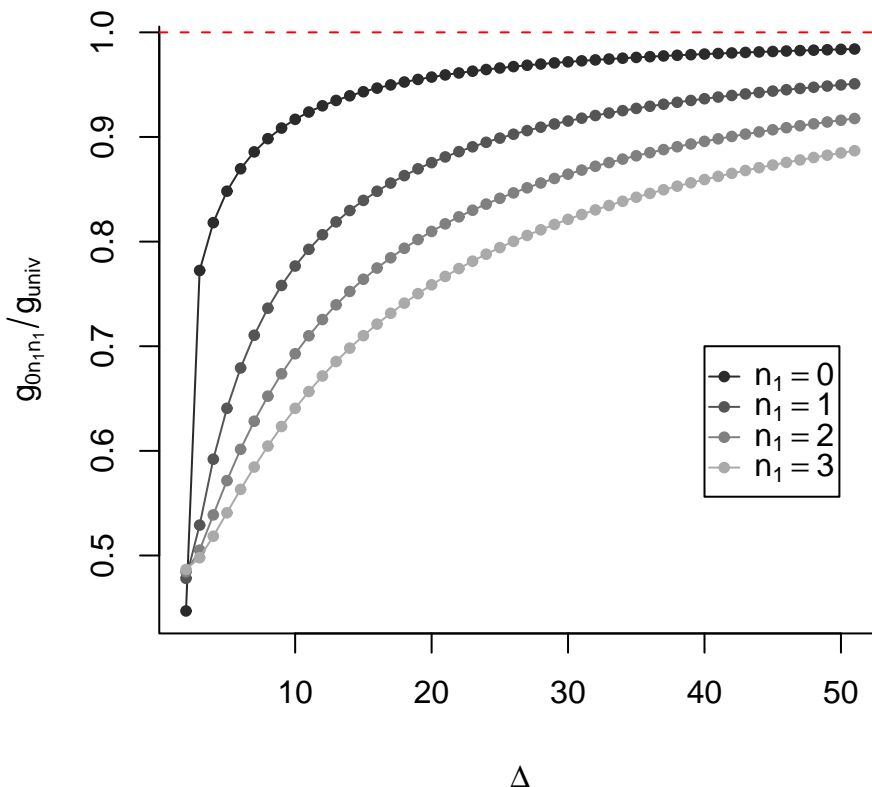


Figure 2: In the hard-wall model, as a function of Δ , the ratio of the coupling g_{0,n_1,n_1}^Δ for scalar hadrons (showing curves for $n_1 = 0, 1, 2, 3$) and its universal value g_{univ} , given in eq. (3.8).

the hard-wall model, in particular, a special property of the Bessel equation; we do not expect this to hold in general models. On the other hand, the fact that g_{0,n_1,n_1}^∞ is *nonzero* is generically the case, since Neumann boundary conditions are required for a conserved current, making $\phi_0(z_{max}) = 0$ unlikely and indeed unnatural. Note, however, that the limit in which eq. (3.8) applies is attained only very slowly as $\Delta \rightarrow \infty$.

In the D3/D7 system, the vector meson coupling to two other mesons of any kind has a universal limit,

$$g_{n,n_1,n_2}^\Delta \xrightarrow{\Delta \rightarrow \infty} \delta_{n_1,n_2} \frac{(2\pi)}{2\sqrt{N}} C_{0,n}^{II} P_n^{(1,1)}(0) = \delta_{n_1,n_2} \frac{(2\pi)}{\sqrt{2N}} \sqrt{\frac{(2n+3)(n+2)}{n+1}} P_n^{(1,1)}(0). \quad (3.9)$$

Again, $g_{n,n_1,n_2}^\Delta \sim 1/\sqrt{N}$ is consistent with N -counting. In this case, we can compare this value with other explicit computations. In [10], $g_{n,0,0}$'s were computed for some specific cases; it can be checked that they have the same limit in the large conformal dimension, which, as we have just argued, is no coincidence. Indeed, using

$$P_n^{(1,1)}(0) = \frac{(n+1)}{2^n} F(-n, -n-1; 2; -1) = \frac{2 \cos \frac{n\pi}{2} \Gamma(\frac{n}{2} + \frac{3}{2})}{\sqrt{\pi} \Gamma(\frac{n}{2} + 2)}$$

we see that eq. (3.9) exactly matches with the limit of eq. (5.7) in [10]. Note that

$\lim_{\Delta \rightarrow \infty} g_{n,n_1,n_2}^\Delta$ vanishes when n is odd. When $n = 2j$ is even,

$$\lim_{\Delta \rightarrow \infty} g_{n,n_1,n_2}^\Delta \xrightarrow{n \rightarrow \infty} \delta_{n_1,n_2} (-1)^j \frac{(2\pi)^j}{\sqrt{N}} \frac{2\sqrt{2}}{\sqrt{\pi}}, \quad (3.10)$$

whose magnitude is n -independent, and differs by only ten percent from

$$\lim_{\Delta \rightarrow \infty} g_{0,n_1,n_2}^\Delta = \delta_{n_1,n_2} \frac{(2\pi)}{\sqrt{N}} \sqrt{3}, \quad (3.11)$$

We have mentioned that the form factors satisfy the power law eq. (3.1) due to the conformal invariance of the field theory in the ultraviolet. In the large conformal dimension case, the power k in eq. (3.1) diverges, so we would expect that in our present approximation this would appear as exponential fall-off at large q^2 . In the hard-wall case, it is not immediately obvious. Given the universal couplings that we have just observed, it would seem that expansion of the form factor as a sum of poles, eq. (2.1), diverges: the coefficient $f_n g_n^\infty \sim \mathcal{O}(n^{3/2})$, where $g_n^\infty \equiv \lim_{\Delta \rightarrow \infty} g_{naa}^\Delta$ is the universal coupling. It is possible to compute the sum by regularizing it carefully, or redefining it by one or more subtractions, but instead we can easily evade the problem altogether. Recalling that the form factor is obtained by the same overlap integral as the tri-meson coupling, eq. (A.5), but with the normalizable mode of the mediating vector meson replaced by a *non-normalizable mode*, we can apply the same approximation to the integral as we did for g_{naa}^Δ . In particular, the large- Δ hadrons of the hard-wall model will have a universal form factor given by the value of the non-normalizable mode at the wall. From eq. (B.3), we use the identity $K_n(x)I_{n+1}(x) + K_{n+1}(x)I_n(x) = 1/x$ at $x = qz_{max} = q/\Lambda$ to obtain

$$F_{ab}(q) = \delta_{ab} \frac{1}{I_0(q/\Lambda)}. \quad (3.12)$$

Indeed it vanishes exponentially at large spacelike q^2 , as we expected.

Similarly, in the D3/D7 case, the universal form factor of the flavor current is given by

$$F_{ab}(\bar{q}) = -\frac{2\pi^{3/2}}{\sin(\pi\alpha)\Gamma(-\frac{\alpha}{2})\Gamma(\frac{1+\alpha}{2})} \quad (3.13)$$

where $\bar{q} = q/m_h$ and $\alpha = (-1 + \sqrt{1 - \bar{q}^2})/2$. This too falls off exponentially at spacelike q^2 .

Highly excited hadrons. Now let's turn to the universality for highly excited hadrons, $a \gg \Delta$. In the hard-wall model, any spin- J normalizable modes, such as Eqs. (B.1) and (B.4), can be approximated as

$$\phi_{p;n}^{(\Delta)}(z) \sim z^{2-J} J_p(\zeta_{p-1;n}\Lambda z) \approx \sqrt{\frac{2}{\pi}} z^{3/2-J} \sin(\zeta_{p-1;n}\Lambda z),$$

for large $\zeta_{p-1;n}\Lambda z$, where p is a constant depending on Δ and J , and $\zeta_{p-1;n}$ is the n -th zero of the Bessel function $J_{p-1}(x)$. The three hadron coupling for a vector hadron $|n\rangle$ and a spin- J hadron $|n_1\rangle$ is

$$g_{n,n_1,n_1}^\Delta = \frac{\sqrt{2}\kappa\Lambda}{R^4} R^{8-2J} \pi^3 \int_0^{1/\Lambda} \frac{dz}{z^{3-2J}} \frac{z J_1(\zeta_{0;n}\Lambda z)}{\pi^{\frac{3}{2}} J_1(\zeta_{0;n})} |\phi_{p,n_1}(z)|^2$$

Since for small z the mode functions are all power-law suppressed, the integral can be approximated using the sine-wave form for the external hadrons, giving the approximately universal coupling

$$\begin{aligned}
 g_{n,n_1,n_1}^\Delta &= \frac{(2\pi)2^{\frac{3}{2}}}{N} \int_0^1 d\hat{z} \frac{\hat{z} J_1(\zeta_{0;n} \hat{z})}{J_1(\zeta_{0;n})} |\sin(m_{p;n_1} \hat{z}/\Lambda)|^2 \\
 &\approx \frac{(2\pi)\sqrt{2}}{N} \int_0^1 d\hat{z} \frac{\hat{z} J_1(\zeta_{0;n} \hat{z})}{J_1(\zeta_{0;n})} \\
 &= \frac{(2\pi)\sqrt{2}}{N} \cdot \frac{\pi}{2\zeta_{0;n}} \mathbf{H}_0(\zeta_{0;n}),
 \end{aligned}$$

where \mathbf{H}_γ is the Struve-H function. In particular,

$$g_{0,n_1,n_1}^\Delta \approx 0.490 \frac{(2\pi)\sqrt{2}}{N}. \tag{3.14}$$

It is interesting to compare this last result with eq. (3.8). The large- Δ and large- n_1 limits do not have the same ρ couplings, and thus the two limits do not commute. However, the couplings in these limits differ only by a factor of about two. Moreover, $g_{0,0,0}^{\Delta=2} = 0.447 \frac{(2\pi)\sqrt{2}}{N}$ for spin-zero hadrons, also quite close to both limits. Indeed, we seem to find that, over the whole domain of Δ and n_i , the couplings of the ρ vary within a rather narrow range. There is no exact universality in this model, but we see no drastic violation of it either.

In the D3/D7 system, we use a similar approximation

$$P_n^{(\alpha,\beta)}(2v-1) = \frac{\cos\left\{[2n+(\alpha+\beta+1)]\cos^{-1}v^{1/2} - \left(\frac{1}{2}\alpha + \frac{1}{4}\pi\right)\right\}}{\sqrt{\pi n}(1-v)^{\alpha/2+1/4}v^{\beta/2+1/4}} + \mathcal{O}(n^{-3/2}). \tag{3.15}$$

This leads to a similar approximation of the overlap integral,

$$g_{n,n_1,n_1}^\ell \approx g_8 \frac{2}{\pi} \int_0^\pi d\theta \phi_n(\theta) |\cos n_1 \theta|^2 \approx \frac{\sqrt{2}(2\pi)}{\sqrt{N}} R^2 \int_0^\pi d\theta \phi_n(\theta) \tag{3.16}$$

where $\cos \theta = 2v - 1$. We can check that eq. (3.16) matches with the form factor computation (5.8) in [10], though this is rather trivial since both results are derived from the same approximation (3.15). The result is

$$\begin{aligned}
 g_{n,n_1,n_1}^\ell &\approx \frac{(2\pi)\sqrt{2}}{\sqrt{N}} \sqrt{\frac{(2n+3)(n+2)}{\pi(n+1)}} \\
 &\quad \times (-1)^n \frac{(n-\frac{1}{2})!}{n!} {}_3F_2\left(\frac{3}{2}, -n, -n-1; 2, \frac{1}{2}-n; 1\right).
 \end{aligned} \tag{3.17}$$

This expression grows as \sqrt{n} for $1 \ll n \ll n_1$. For the ρ , the $n = 0$ case, we have

$$g_{0,n_1,n_1}^\ell \approx \frac{(2\pi)}{\sqrt{N}} \sqrt{3}. \tag{3.18}$$

Note that Eqs. (3.10) and (3.17) differ, although for the ρ meson, interestingly, the large- Δ and large- n_1 limits give the same result, Eqs. (3.11) and (3.18). We will see this can be viewed as resulting from the exact universality that we discuss in the next section.

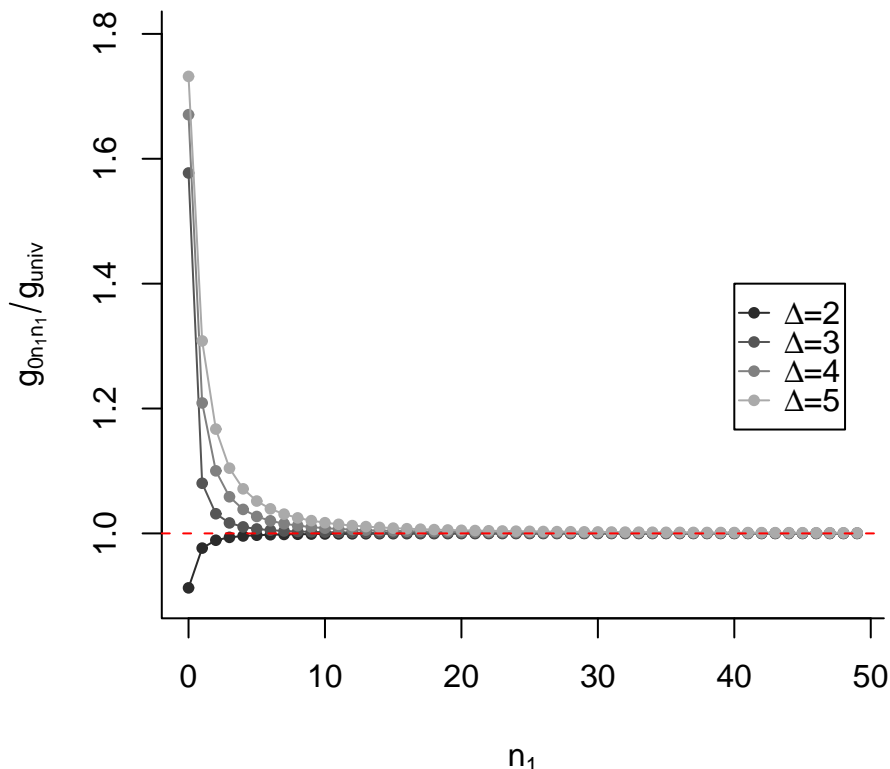


Figure 3: In the hard-wall model, as a function of n_1 , the ratio of the coupling g_{0,n_1,n_1}^Δ for scalar hadrons and its universal value g_{univ} , given in eq. (3.14).

Also, (3.18) is of the same order as the couplings of the ρ to the lowest-lying mesons in the theory. In particular, $g_{000}^{\ell=1}$ for external scalar hadrons is given in eq. (2.10), $g_{000}^{\ell=1} = \frac{(2\pi)}{\sqrt{N}} \frac{\sqrt{3}}{2}$, while $g_{000}^{\ell=0}$ for external vector hadrons, from eq. (2.16), is again $g_{000}^{\ell=0} = \frac{(2\pi)}{\sqrt{N}} \sqrt{3}$.

As was the case for large Δ , the universal form of the couplings implies a universal form factor. Applying our approximation strategy for high excitation modes to the form factor calculation, we find a universal form factor for highly excited hadrons in the hard-wall model,

$$\lim_{n_1 \rightarrow \infty} F_{n_1, n_1}(q) = \frac{\pi}{2} \mathbf{L}_0(q/\Lambda) \left[K_1(q/\Lambda) + \frac{K_0(q/\Lambda)}{I_0(q/\Lambda)} I_1(q/\Lambda) \right] \quad (3.19)$$

where \mathbf{L}_γ is the Struve-L function. Similarly, highly excited hadrons in the D3/D7 model have the universal form factor

$$\lim_{n_1 \rightarrow \infty} F_{n_1, n_1}(q) = \frac{4}{\bar{q}^2} - \frac{\pi}{\cos\left(\frac{\pi}{2} \sqrt{1 - \bar{q}^2}\right)} \quad (3.20)$$

where $\alpha = (-1 + \sqrt{1 - \bar{q}^2})/2$.

Additional comments. In some of the above examples, the g_{naa} seem to exhibit n -independence, or even growth with n , for large n . This behavior must break down, because of the power law (3.1). From (2.1), the power law can only hold if the moments $\sum f_n g_{nab} m_n^{2j}$, for all $j < k$ (where k is the power in eq. (3.1)), vanish. Truly universal

a -independent and/or Δ -independent couplings g_{naa}^Δ , for high but fixed a, Δ and for all n , would endanger this power law. Consequently, any universality with respect to n must break down eventually. If n, a and b are all very large, the computation of g_{nab} involves the overlap integration of a product of three rapidly oscillating functions, and for sufficiently large n this will begin to decrease. Similarly, when $n \gg \Delta$ the spin-one mode ϕ_n oscillates so quickly that one cannot treat the external hadron as localized on the scale of the oscillations.

An interesting pattern which appears in both models concerns the couplings of the ρ . Along with all the other vector meson couplings, $g_{\rho aa}^\Delta$ has a universal value at large Δ , and a second universal value at large a . These differ, but are of the same order, in the hard-wall model; in the D3/D7 model the two limits commute, for reasons that we will see in the next section. In all cases the coupling in these limits differ from the ρ -dominance prediction m_ρ^2/f_ρ , given in Eqs. (2.8) and (2.14), but only by a factor of order two. Finally, neither differs much from the (non-universal) couplings of the ρ to the lowest-lying mesons in the theory, including its own self-coupling. In short, we do not find that the conjecture of universal couplings is true, but neither do we find it badly violated. This deserves an explanation, which none of the arguments presented in this paper directly provides. We will address this issue further in a future publication [27].

3.3 Exact universality

Amusingly, we have found one example of exact universality for the couplings of the ρ to a certain class of hadrons. As hinted already by some of our earlier calculations, this arises in a subsector of the D3/D7 system. The universality can be derived from a certain symmetry satisfied by the relevant mode functions, but we have not found that this mathematical property of the modes has any deeper physical significance. A similar sector in the hard-wall model does not show exact universality. At this level, then, the example we now present appears special to this model, and in this sense, accidental.

We begin with the type II modes, of which the ρ is one, which lie within the subsector exhibiting universality. The coupling of three type II modes is computed using (2.15). One portion of the integrand involves two mode functions and a metric factor

$$\left(\frac{1-v}{v}\right)\phi_{\ell,n_1}^{II}\phi_{\ell,n_2}^{II} = R^{-4}\hat{C}_{\ell n_1}^{II}\hat{C}_{\ell n_2}^{II}v^{\ell+1}(1-v)^{\ell+1}P_{n_1}^{(\ell+1,\ell+1)}(2v-1)P_{n_2}^{(\ell+1,\ell+1)}(2v-1). \tag{3.21}$$

This is invariant under the transformation $v \rightarrow 1-v$, up to the sign $(-1)^{n_1+n_2}$. The remaining factor of the integrand is the wave function $\phi_{0,n}^{II}$, which transforms non-trivially under $v \rightarrow 1-v$, or equivalently $\varrho \rightarrow 1/\varrho$. We can decompose this function into the sum of odd and even parts. The higher is n , the more complicated is each part, but for the ρ meson, the lowest mode $n=0$, the wave function is very simple:

$$\phi_{0,0}^{II} = R^{-2}\hat{C}_{0,0}^{II}v = R^{-2}\hat{C}_{0,0}^{II}\left[\frac{1}{2} + \left(v - \frac{1}{2}\right)\right].$$

If $n_1 - n_2$ is even, then the odd part of this function can be dropped; the even part is constant and the computation reduces to the normalization integral of the modes. Therefore, we

find that within this sector the ρ has a universal diagonal coupling

$$g_{\rho,n_1,n_1}^\ell = \frac{(2\pi)}{\sqrt{N}}\sqrt{3}. \tag{3.22}$$

This result extends beyond the type II modes, due to an accidental symmetry relating the type II mode to others. It has been found that the D7 brane worldvolume theory has extra degeneracy among the scalar, type II and type III modes. This degeneracy is not required by any obvious symmetry, but its presence has been interpreted as a sign of an extension to $SO(5)$ of the explicit $SO(4)$ present in the classical field theory [18]. We will refer to the degenerate modes as the “ $SO(5)$ multiplet.” Note that the $SO(5)$ symmetry relates states with different spin. For example, it relates the pion-like scalar mesons created by $\psi_Q^\dagger \Phi^\ell \psi_Q$, and the ρ -like spin-one mesons. All the modes in the $SO(5)$ multiplet are eigenvectors of the transformation $v \rightarrow 1 - v$, which as we have just seen above leads them to have the universal coupling (3.22) to the ρ , and indeed, to all of the ground states related to the ρ by $SO(5)$.

It is easy to check that the scalar and the type III modes have the same behavior and universal coupling as those of type II. For the scalar mode, the product of the wavefunctions and the metric factor gives exactly eq. (3.21), just as for type II, so the same universality is trivially obtained. The type III modes are different in appearance, as they involve gauge fields polarized both in the compact S^3 directions and the fifth dimension, but in the end the integral is the also the same as for type II.

From (3.22), it follows that the large dimension limit $\ell \rightarrow \infty$ and the large excitation limit $n_1 \rightarrow \infty$ lead to the same limiting ρ coupling in these particular sectors. But the coupling arising in each limit is the same in *all* sectors. Therefore, in all sectors, the ρ coupling must approach (3.22) both at large dimension and at large excitation. This explains why Eqs. (3.11) and (3.18) agree with each other and with (3.22).

As we noted, the key fact leading to universal couplings is that the integrands in Eqs. (A.2) and (A.8) are identical except for the mode function ϕ_n of the spin-one vector meson. One might ask if there are other natural contexts where symmetries might constrain ϕ_n , or in particular the ρ mode function ϕ_0 , such that the overlap computation would reduce to the normalization integral (A.2), giving a universal value for the ρ 's coupling to all hadron states. In the D3/D7 case above, the “parity” $v \rightarrow 1 - v$ (really an inversion symmetry $\varrho \rightarrow 1/\varrho$) played such a role. It would be interesting to build this feature into a model to obtain the universality seen in QCD, something along the phenomenological lines of [8]. We leave this question for future study.

Examples

We already have computed in section 2 one example of a tri-meson coupling in the type II sector, eq. (2.16). Letting $n = n_2$, and using the cyclic symmetry of the coupling (2.15), $g_{0,n,n}^\ell = g_{n,0,n}$, we see that $g_{0,0,0}^0$ is indeed the exactly universal coupling $\frac{(2\pi)}{\sqrt{N}}\sqrt{3}$. Note however that m_ρ^2/f_ρ , eq. (2.14), is smaller by a factor of $\frac{3}{2}$.

4. Conclusions and discussion

We have examined the universality of the ρ meson's couplings, and those of excited vector mesons, in the AdS/CFT context. We did not find that the ρ typically has precisely universal couplings. We did find two regimes of approximate coupling-universality, which become exact in certain limits. These are especially interesting because they are generic, arising for fundamental reasons which apply in any theory at large 't Hooft coupling. The first case is the ρ 's couplings (and those of other vector mesons) to hadrons created by interpolating operators of very large conformal dimension; in this case, universality stems from the localization of the associated mode functions at the minimum of an appropriate effective potential. The second case involves hadrons which are highly excited states; the wave functions oscillate rapidly, and these fluctuations average out in the calculation of the couplings to vector mesons. Note that in general the two different universalities do not commute with each other, indicating that they represent two distinct regimes. Moreover, as we saw examining two models, the ρ 's couplings do not match the conjectured value m_ρ^2/f_ρ in either regime, though they do not differ from it by more than a factor of two in either model.

We also saw that a large sector of one model (the D3/D7 system) exhibits exact coupling-universality for the ρ . As a consequence, the two above-mentioned limits commute in this model. This feature requires special properties which constrain the mode function of the ρ relative to the other modes. We expect this behavior is highly model-dependent and does not generically arise elsewhere.

For further study, then, there are two main questions that we should ask. First, why are the ρ couplings often roughly universal, and over what range can they vary in generic models? Clearly there is a connection with the fact that the ρ is created by a conserved current, which has the special property that its non-normalizable mode at $q^2 = 0$ is always a constant in the radial direction, with a fixed normalization, in order to ensure $F(q^2 \rightarrow 0) \rightarrow 1$. Second, why does $f_\rho g_{\rho aa}$ tend to be of order m_ρ^2 *even when* many other vector mesons are large contributors to a form factor? This, too, is presumably tied to the particular shape of the ρ meson's mode function, which, being generally positive definite and structureless, is significantly constrained. We leave these questions for further study [27].

There are several other interesting problems which were not dealt with in this paper. One is the computation of corrections beyond supergravity. There have been many interesting approaches to this problem [15, 20–23]. In particular, it has been noted that the string theory can be highly simplified for large conformal dimension $\Delta \sim \sqrt{N}$ [22], where one of our examples of universality arises. Aided by these simplifications, this limit may serve as a nice testing ground to discover more interesting relationships between string theory and QCD.

The conjecture of universal couplings includes nucleons as well. We did not consider baryons here, as at large N they are very different objects from mesons. Indeed, at large 't Hooft coupling they are described by D-branes rather than of supergravity modes. At present there is no suitable tool for the relevant computations; the baryons' charges can

be calculated using geometry, but the formalism for computing dynamic quantities such as a form factor is still undeveloped. Still, the baryons are localized at small radius in much the same way as $\Delta \gg 1$ mesons, and for much the same reason. We might therefore expect that they share the same universal form factor as large- Δ mesons, but this remains to be confirmed. For five-dimensional states with extremely large mass (corresponding to field-theoretic operators of dimension much larger than N) back-reaction on the metric eventually becomes important. This back-reaction would be relevant, for instance, for nuclei with large numbers of baryons. Although these objects, too, tend to localize due to their heavy mass, their backreaction on the metric is likely to alter their form factors.

One may consider the experimental implications of these findings, but a little thought reveals the situation is not encouraging for any direct application. It is very difficult to measure tri-vector-meson couplings, even $g_{\rho\rho\rho}$, or form factors of unstable particles, even the ρ . As we have discussed and have seen in the examples, there is no reason to believe that $g_{\rho\rho\rho}$ is approximately equal to, for example, $g_{\rho\pi\pi}$, at least in the large λ limit. Indeed our examples suggest that $g_{\rho\rho\rho}$ can differ from $g_{\rho\pi\pi}$ by a factor of 2 or so. An attempt could be made to measure $g_{\rho\rho\rho}$ in the process $\pi^+ p \rightarrow \pi^+ \rho^+ n$, but to extract $g_{\rho\rho\rho}$ in a fully model-independent way would not be possible in this experiment; one would have to assume ρ dominance in the intermediate states. Meanwhile, the approximate universalities that we found for certain states in the AdS/CFT models are completely out of experimental reach. Perhaps there are more subtle ways to apply our results to QCD, but we will have to seek them in the future.

Still, it is interesting to observe that although most of our results are, in a sense, negative, in that we do not confirm the classic conjectures, we still have the unexplained fact that the ρ couplings to most objects in the theory appears to be of the same order. The structure of the calculation in AdS/CFT seems to suggest that this arises from profound properties of mesons created by conserved currents. In this sense, Sakurai's original idea of treating the ρ as a gauge boson seems not entirely misguided. We will return to this issue in [27].

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A. Review of methodology

The methodology we use in this paper is established originally in the hard wall model [13–15] and applied to the the D3/D7 system [10].

We first assume that our confining model is given by the asymptotically d -dimensional $\text{AdS}_5 \times W$ space (W a compact manifold of dimension $d-5$) which has the following metric

$$ds^2 = \frac{R^2 e^{2A(z)}}{z^2} (\eta_{\mu\nu} dx^\mu dx^\nu + dz^2) + \hat{g}_{\perp ij} d\hat{z}^i d\hat{z}^j, \quad (\text{A.1})$$

where $e^{2A(z)} \rightarrow 1$ as $z \rightarrow 0$. x^μ is tangential to the four dimensions, z the AdS radius, and \hat{z}^i 's are the coordinates on W . $\hat{g}_{\perp ij}$ is the metric of W . We assume that the square of the warp factor $e^{2A(z)}/z^2$ has a minimum at $z = z_{max}$. This is one of the sufficient conditions that this background is dual to a confining gauge theory [26]. Also, in the light-cone gauge, the warp factor squared has a natural interpretation as the potential for a classical long string [14] in the string action. Therefore, z_{max} can be interpreted as “the wall at the end of space” [26], beyond which a string cannot go.

Each hadron state is dual to a *normalizable* mode in five dimensions. When a spin- J ($J \leq 2$) field is given by $C_{\mu_1 \mu_2 \dots \mu_J} = \epsilon_{\mu_1 \mu_2 \dots \mu_J} e^{ik \cdot x} \phi(z) \mathcal{Y}^\ell(W)$, then the normalizable mode $\phi_n(z)$ with $k^2 = -m_n^2$ satisfies the normalization condition

$$R^{3-2J} \int_0^{z_{max}} \frac{dz}{z^{3-2J}} e^{(3-2J)A(z)} V(z) \phi_{n_1} \phi_{n_2} = \delta_{n_1 n_2}, \quad (\text{A.2})$$

where $V(z)$ is a normalization coefficient in W direction

$$V(z) = \int d^{d-5} \hat{z} \sqrt{g_{\perp}} \left| \mathcal{Y}^\ell(W) \right|^2. \quad (\text{A.3})$$

In principle, we might encounter hadron states dual to bulk vector or rank-two tensor fields which are partially or entirely polarized in the \hat{z}^i directions. In such cases, we would have to include suitable \hat{g}^{ij} factors in the integrand of eq. (A.2). However, we can absorb such factors into the wavefunctions, and treat such fields as five dimensional scalar or vector fields satisfying eq. (A.2). Also note that $V(z)$ depends on the normalization of $\mathcal{Y}^\ell(W)$ on W , which can be arbitrarily chosen. In this paper, we use the convention that the norm of $\mathcal{Y}^\ell(W)$ is equal to the volume of W . In particular, this sets the lowest constant mode $\mathcal{Y}^0(W) = 1$, and $V(z)$ is the volume of W at z .

To compute the matrix element of a current, we need the *non-normalizable* mode dual to that current. Then we find the trilinear interaction between the three modes corresponding to the initial state, the final state and the current operator. Such an interaction can be derived either from bulk supergravity or the Born-Infeld action on D7 branes if present. The matrix element for the spin- S current and the spin- J hadrons is given by

$$\langle b | \mathcal{J}^{\mu_1 \mu_2 \dots \mu_S} | a \rangle = (\text{charge}) \times (\text{kinematic factor}) \times F_{ab}(q^2), \quad (\text{A.4})$$

where the form factor $F_{ab}(q^2)$ is⁵

$$F_{ab}(q^2) = g_d R^{5-2(S+J)} \int \frac{dz}{z^{5-2(S+J)}} e^{(5-2(S+J))A(z)} V(z) \psi(q, z) \phi_a \phi_b. \quad (\text{A.5})$$

⁵Here we again ignore that we need multiple form factors depending on J and whether \mathcal{J} is conserved as discussed in 2.

We denoted the non-normalizable mode for the current operator by ψ , the normalizable modes by $\phi_{a,b}$ and the suitable five-dimensional coupling constant by g_d .

The form factor must satisfy a constraint $F_{ab}(q^2 = 0) = \delta_{ab}$. In our context, this is related to the proper normalization of the non-normalizable mode ψ . When the gauge theory is conformal, so that the five-dimensional spacetime is AdS_5 , the usual choice of normalization in the AdS/CFT context is

$$\lim_{q^2 \rightarrow 0} (z/R)^{2(S-1)} \psi(q^2, z) = \lim_{z \rightarrow 0} (z/R)^{2(S-1)} \psi(q^2, z) = 1. \quad (\text{A.6})$$

For this reason, we present the non-normalizable modes in this paper with the similar normalization,

$$\lim_{q^2 \rightarrow 0} (z/R)^{2(S-1)} e^{-2(S-1)A(z)} \psi(q^2, z) = 1, \quad (\text{A.7})$$

which reduces to eq. (A.6) in the ‘‘conformal limit’’ $z \rightarrow 0$. However, whenever we use these modes in the computation of form factors, we need to make them ‘‘canonically normalized.’’ This is accomplished by scaling $\psi \rightarrow \psi/g_d$, which ensures that eq. (A.5) reduces to eq. (A.2) in the $q^2 \rightarrow 0$ limit, and that $F_{ab}(q^2 = 0) = 1$.

We can compute another quantity, which corresponds to a hadron coupling constant among three hadron states. It is the three hadron overlap, obtained in the following way. When the hadron states are labeled by n , a and b , the three hadron coupling is given by

$$g_{nab} = g_d R^{5-2(S+J)} \int \frac{dz}{z^{5-2(S+J)}} e^{(5-2(S+J))A(z)} V(z) \varphi_n \phi_a \phi_b. \quad (\text{A.8})$$

B. Review of the hard-wall and the D3/D7 model

Here we add a brief explanation of the models that we used for examples and list the mode functions.

As explained earlier, the hard-wall model is given by the $AdS_5 \times S^5$ space with a wall at a finite radius. This wall puts the boundary condition the mode functions and we choose the Neumann condition. The metric is just given by the $AdS_5 \times S^5$ metric, $A(z) = 0$ in Eq (A.1), up to the location of the wall, $z = z_{max} = 1/\Lambda$. Then the spin one normalizable mode corresponding to the current operator is given by

$$\begin{aligned} A_\mu(m_n) &= \epsilon_\mu \phi_n(z) \mathcal{Y}^0(S^5), \quad \mathcal{Y}^0(S^5) = 1, \\ \phi_n(z) &= \frac{\sqrt{2}z/z_{max} J_1(\zeta_{0;n}z/z_{max})}{\pi^{\frac{3}{2}} R^3 J_1(\zeta_{0;n})}, \end{aligned} \quad (\text{B.1})$$

where $\zeta_{k;n}$ denotes the n -th zero of the Bessel function $J_k(x)$. We also expressed the mode in terms of the v coordinate that we introduce below for the D3/D7 system. The mass is

$$m_n = \zeta_{0;n} \Lambda \xrightarrow{n \gg 1} \left(n - \frac{1}{4} \right) \pi \Lambda. \quad (\text{B.2})$$

The corresponding non-normalizable mode is

$$\begin{aligned} \hat{A}_\mu(m_n) &= \epsilon_\mu \psi(q, z), \\ \psi(q, z) &= qz \left\{ K_1(qz) + \frac{K_0(q/\Lambda)}{I_0(q/\Lambda)} I_1(qz) \right\} \Big|_{\Lambda \rightarrow 0} \approx qz K_1(qz). \end{aligned} \quad (\text{B.3})$$

Canonical normalization for this mode requires dividing by $g_d = g_{10} = \kappa/R$, where $\kappa^2 = (2\pi)^7 \alpha'^4 g_s^2/2$ and $R^4 = 4\pi g_s N \alpha'^2$; thus $g_{10} = 2\pi^{5/2} R^3/N$. The volume of the internal manifold is that of a 5-sphere of constant radius R : $V(z) = \pi^3 R^5$.

The spin-two case is similar. The normalizable and non-normalizable modes are

$$h_{\mu\nu} = \epsilon_{\mu\nu} \frac{\sqrt{2} J_2(\zeta_{1,n} z/z_{max})}{\pi^{\frac{3}{2}} z_{max} R^2 J_2(\zeta_{1;n})} \mathcal{Y}^0(S^5), \tag{B.4}$$

$$\hat{h}_{\mu\nu} = \epsilon_{\mu\nu} \frac{R^2 q^2}{2} \left\{ K_2(qz) + \frac{K_1(q/\Lambda)}{I_1(q/\Lambda)} I_2(qz) \right\} \mathcal{Y}^0(S^5). \tag{B.5}$$

For a scalar hadron created by an operator with conformal dimension Δ , we use the following mode

$$\phi_n^{(\Delta)}(z) = \frac{\sqrt{2} z^2 J_{\Delta-2}(\zeta_{\Delta-3;n} z/z_{max})}{\pi^{\frac{3}{2}} R^4 z_{max} J_{\Delta-2}(\zeta_{\Delta-3;n})}, \tag{B.6}$$

which satisfies the boundary condition

$$\partial_z \left[z^{\Delta-4} \phi_n^{(\Delta)}(z) \right] \Big|_{z=z_{max}} = 0,$$

which is analogous to Neumann condition for the vector and rank two tensor modes.

The D3/D7 model is described in detail in [16–18, 10]. Here we summarize only what is needed in the computations. The theory is composed of two sectors, $\mathcal{N} = 4$ $SU(N)$ Yang-Mills theory and N_f of $\mathcal{N} = 2$ hypermultiplets which are in fundamental representation of $SU(N)$. It has the global symmetry $SO(4) \approx SU(2) \times SU(2)$ symmetry, consisting of an $SU(2)_\Phi$ symmetry rotating Φ_1 and Φ_2 and an $SU(2)_R$ $\mathcal{N} = 2$ R-symmetry. The superpotential is

$$W = \sqrt{2} \operatorname{tr}([\Phi_1, \Phi_2]\Phi_3) + \sum_{r=1}^{N_f} Q^r \Phi_3 \tilde{Q}_r + m_r Q^r \tilde{Q}_r$$

where m_r is the mass of hypermultiplet r and the trace is over color indices. If all the masses m_r are equal, as we will assume throughout, there is additional flavor symmetry $SU(N_f)$.

For large $g^2 N$, and in the “quenched limit” $N_f \ll N$, the theory is dual to IIB supergravity in $AdS_5 \times S^5$ with N_f probe D7 branes [16]. The induced metric on the D7 brane is given by

$$\begin{aligned} ds^2 &= \frac{r^2}{R^2} \eta_{\mu\nu} dx^\mu dx^\nu + \sum_{c=4}^7 \frac{R^2}{r^2} (dx^c)^2 \\ &= \frac{L^2}{R^2} (\varrho^2 + 1) \eta_{\mu\nu} dx^\mu dx^\nu + R^2 \frac{1}{\varrho^2 + 1} d\varrho^2 + R^2 \frac{\varrho^2}{\varrho^2 + 1} d\Omega_3^2, \end{aligned} \tag{B.7}$$

where $\varrho^2 = \frac{r^2}{L^2} - 1$, and the S^3 involves the angular coordinates in the four-dimensional space spanned by x^4, x^5, x^6, x^7 .

All of the Born-Infeld modes on the D7 brane were exactly calculated in [18], where the modes were classified as scalar, I \pm , II and III. There are extra degeneracies, not explained

by the explicit global symmetries, among the scalar, II and III modes. This signifies the existence of an extended $SO(5)$ accidental symmetry, containing the original $SO(4)$ inside.

In [10], a set of coordinates was introduced such that the expressions for the mode functions become more convenient for form factor computation than those presented in [18]. These the coordinates are

$$v = (L/r)^2, \quad w = 1 - v; \quad \varrho^2 = v^{-1} - 1 = \frac{w}{1 - w}. \quad (\text{B.8})$$

Among the various quarkonium modes in the theory, we only present the mode I- and II, which we mainly used in this paper. The detailed discussion on other modes can be found in [18].

Type I-: the I- modes correspond to the D7 brane worldvolume Yang-Mills field polarized in the S^3 directions. Using the global charges and the conformal dimension, they can be uniquely identified as dual to the operators

$$(\tilde{Q}\Phi^{\ell-1}Q)_{\theta, \bar{\theta}=0} = \tilde{Q}\Phi^{\ell-1}Q + \dots \quad (\text{B.9})$$

The masses of the normalizable mode are

$$M_{I-}^2 = 4m_h^2(n + \ell)(n + \ell + 1).$$

The wavefunctions are

$$A_\mu = 0, \quad A_\rho = 0, \quad A_\alpha = \phi^{I-}(\rho)e^{ik \cdot x} \mathcal{Y}_\alpha^{\ell,-}(S^3) \quad (\text{B.10})$$

$$\begin{aligned} \phi_{\ell,n}^{I-} &= (C_{\ell n}^I/L)\varrho^{\ell+1}(1 + \varrho^2)^{-1-n-\ell} F(-n, 1 - n - \ell; \ell + 2; -\varrho^2) \\ &= (\hat{C}_{\ell n}^I/L)v^{(\ell+1)/2}(1 - v)^{(\ell+1)/2} P_n^{(\ell+1, \ell-1)}(2v - 1) \end{aligned}$$

where $n \geq 0$, $\ell \geq 1$, and

$$C_{\ell n}^I = \frac{1}{\pi} \sqrt{(2n + 2\ell + 1) \binom{n + 2\ell}{\ell + 1} \binom{n + \ell + 1}{\ell + 1}} = \binom{n + \ell + 1}{\ell + 1} \hat{C}_{\ell n}^I.$$

Type II: these modes correspond to the worldvolume gauge field polarized in 0123 directions, and are dual to the flavor current operator and its generalizations,

$$(Q^\dagger \Phi^\ell Q - \tilde{Q} \Phi^\ell \tilde{Q}^\dagger)_{\theta \bar{\theta}} = Q^\dagger \Phi^\ell \partial^\mu Q + \psi_Q^\dagger \Phi^\ell \sigma^\mu \psi_Q - \tilde{Q} \Phi^\ell \partial^\mu \tilde{Q}^\dagger + \dots \quad (\text{B.11})$$

where Φ^ℓ stands for any product of Φ_1 and Φ_2 which is a symmetric and traceless representation under $SO(4)$. The masses of the normalizable mode are

$$M_{II}^2 = 4m_h^2(n + \ell + 1)(n + \ell + 2),$$

The wavefunctions are

$$A_\rho = 0, \quad A_\alpha = 0, \quad A_\mu = \zeta_\mu \phi^{II}(\rho) e^{ik \cdot x} \mathcal{Y}^\ell(S^3), \quad k \cdot \zeta = 0 \quad (\text{B.12})$$

$$\begin{aligned}\phi_{\ell,n}^{II} &= (C_{\ell n}^{II}/R^2)\varrho^\ell(1+\varrho^2)^{-1-n-\ell}F(-n,-1-n-\ell;\ell+2;-\varrho^2) \\ &= (\hat{C}_{\ell n}^{II}/R^2)v^{(\ell+2)/2}(1-v)^{\ell/2}P_n^{(\ell+1,\ell+1)}(2v-1),\end{aligned}$$

where $n \geq 0$, $\ell \geq 0$,

$$C_{\ell n}^{II} = \frac{1}{\pi} \sqrt{(2n+2\ell+3) \binom{n+2\ell+2}{\ell+1} \binom{n+\ell+1}{\ell+1}} = \binom{n+\ell+1}{\ell+1} \hat{C}_{\ell n}^{II},$$

and $P_n^{(\alpha,\beta)}(x)$ denotes a Jacobi polynomial. In addition to the normalizable mode, the non-normalizable mode of type II is dual to the flavor current operator. It is

$$\psi^{II}(q) = \frac{\pi\alpha(\alpha+1)}{\sin\pi\alpha} {}_2F_1(-\alpha, \alpha+1, 2, 1-v) \quad (\text{B.13})$$

where $\alpha = (-1 + \sqrt{1 - (q/m_h)^2})/2$. Again, canonical normalization requires dividing by $g_d = g_8$, the Yang-Mills coupling in eight dimensions: $g_8 = (2\pi)^{5/2} g_s^{1/2} \alpha' = 2\sqrt{2}\pi^2 R^2 / \sqrt{N}$. The volume of the 3-sphere is $V(w) = 2\pi^2 R^3 w^{3/2} = 2\pi^2 R^3 (1-v)^{3/2}$.

C. Proof of decomposition formula

Assume that the spacetime is asymptotically $\text{AdS}_5 \times W$, with metric is given as eq. (A.1). We consider a five-dimensional gauge field $C_\mu = \epsilon_\mu e^{iq \cdot x} \chi(q^2, z)$, where z is the five-dimensional radial coordinate defined in appendix A. The normalizable mode is $\phi_n(z) \propto \chi(-m_n^2, z)$ at $q^2 = -m_n^2$, and the non-normalizable mode is $\psi(q^2, z) \equiv \chi(q^2, z)$ for arbitrary q^2 . With gauge $C_z = 0$, $q \cdot C = 0$, the action for $\psi(q^2, z)$ is

$$S = \int_0^{z_{max}} dz \left(\frac{R e^{A(z)}}{z} \right) V(z) [(\partial_z \psi)^2 + q^2 \psi^2],$$

which gives us the equation of motion

$$\mathcal{L}\psi - q^2\psi = 0, \quad \mathcal{L} = \frac{e^{-A(z)} z}{R V(z)} \partial_z \left(\frac{R e^{A(z)}}{z} V(z) \partial_z \right).$$

Now the problem effectively reduces to that of a field in a one-dimensional cavity. The linearity of the equation allows us easily to obtain Green's theorem:

$$\begin{aligned}\int_0^{z_{max}} dz \left(\frac{R e^{A(z)}}{z} \right) V(z) [\psi(\mathcal{L} - q^2)\chi - \chi(\mathcal{L} - q^2)\psi] \\ = - \lim_{z' \rightarrow 0} [\psi(z') \mathcal{D}_{z'} \chi(z') - \chi(z') \mathcal{D}_{z'} \psi(z')], \quad (\text{C.1})\end{aligned}$$

where

$$\mathcal{D}_{z'} = \frac{R e^{A(z')}}{z'} V(z') \partial_{z'}.$$

We assumed here that there is no additional source at $z = z_{max}$, which is automatically guaranteed by a Neumann boundary condition for the gauge field at $z = z_{max}$. This implies that our solution $\psi(q^2, z)$, with Neumann boundary conditions at $z = 0$, can be obtained

as

$$\psi(q^2, z) = \psi(q^2, 0) \lim_{z' \rightarrow 0} \frac{R}{z'} V(z') \partial_{z'} G(z, z'; q^2) \quad (\text{C.2})$$

where we used $e^{A(z)} \rightarrow 1$ as $z \rightarrow 0$ in the asymptotically AdS space. $G(z, z'; q^2)$ is the Green's function satisfying the equation

$$\left(\frac{R e^{A(z)}}{z} \right) V(z) (\mathcal{L} - q^2) G(z, z'; q^2) = -\delta(z - z'), \quad (\text{C.3})$$

with Dirichlet boundary conditions. Since the normalizable modes form a complete basis, we can construct the Green's function as

$$G(z, z'; q^2) = \sum_n \frac{\phi_n(z) \phi_n(z')}{q^2 + m_n^2}. \quad (\text{C.4})$$

It can be easily checked that eq. (C.4) satisfies (C.3) by using the completeness relation,

$$\left(\frac{R e^{A(z)}}{z} \right) V(z) \sum_n \phi_n(z) \phi_n(z') = \delta(z - z').$$

Hence, with ‘‘canonical normalization’’⁶ $\psi(q^2, z = 0) = 1/g_d$, we obtain eq. (2.2) with (2.3) by plugging (C.4) in (C.2). The generalization to the other spin cases is straightforward and again yields eq. (2.2). Therefore, the decomposition (2.1) is exact for every conserved current in the large λ limit.

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